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Canonical Decomposition of the General Relativistic Initial Value Problem

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A canonical transformation is employed to implement a conformal transformation of the configuration variables of general relativity. The transformation is so chosen that the spatial constraints become algebraic in the trace of the momentum density. The temporal constraint is then found to have the form of York and O'Murchadha. The role played by the York coordinate condition in decoupling the constraint equations is examined, and a procedure to solve the constraint equations without employing such a coordinate condition is sketched.

1. INTRODUCTION

In the Dirac canonical formalism (Dirac, 1959) the initial value problem for the general theory of relativity is stated by giving a pair of canonically conjugate symmetric fields over a three-dimensional manifold, g_{ij}, p^{ij}, subject to the constraint equations

$$\mathscr{H}^s \equiv -2p^{sm}|_m = -2(p^{sm}|_m + p^{mn}\Gamma^s_{mn}) = 0 \tag{1.1}$$

$$\mathscr{H}_L \equiv g^{-1/2}(p^{mn}p_{mn} - \tfrac{1}{2}p^2) + g^{1/2}R = 0 \tag{1.2}$$

where g_{ij} is regarded as a Riemannian metric for the purpose of raising and lowering indices, forming covariant derivatives (denoted by the bar subscript), forming the affine connection Γ^s_{mn}, forming the curvature scaler R, forming the trace of the canonical momentum tensor density p, and where g is the determinant of g_{ij}. A central problem both for classical gravitation theory and for the development of a quantum theory of gravitation is the solution of these constraints. In a recent series of papers (York, 1972, 1973; O'Murchadha and York, 1973) a recursive procedure was developed to exhibit the existence of general solutions of these equations. In brief, if one

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imposes the coordinate condition upon the selection of the constant time surfaces in the four-dimensional space-time

$$t = \tfrac{2}{3} g^{-1/2} p \tag{1.3}$$

on the initial surface equation (1.1) can as well be written

$$(p^{sm} - \tfrac{1}{3} p g^{sm})_{|m} = 0 \tag{1.4}$$

The virtue of writing the spatial constraints in this form is that it is invariant under the conformal transformation of the metric tensor

$$\bar{g}_{ij} \equiv \varphi^4 g_{ij} \tag{1.5}$$

provided the canonically conjugate tensor density p^{ij} is understood to transform simultaneously via

$$\bar{p}^{ij} \equiv \varphi^{-4} p^{ij} \tag{1.6}$$

Equation (1.4) may be solved for a canonically conjugate pair g_{ij}, p^{ij} in a variety of ways (York, 1972, 1973). If one then selects φ to be a solution of the equation

$$8\varphi_{|m}^{|m} = R\varphi + M\varphi^{-7} - \tfrac{3}{8} t^2 \varphi^5 \tag{1.7}$$

where

$$M \equiv g^{-1}(p^{mn} p_{mn} - \tfrac{1}{3} p^2) \tag{1.8}$$

it can easily be confirmed that the canonically conjugate pair $\bar{g}_{ij}, \bar{p}^{ij}$ defined by equations (1.5) and (1.6) will satisfy the constraint equations (1.2) as well as (1.1). In addition, it has been proven (O'Murchadha and York, 1973) that for $M > 0$, $t \neq 0$, a solution for equation (1.7) such that $0 < \varphi < \infty$ everywhere on a closed, or asymptotically flat open, three-dimensional manifold exists and is unique (modulo boundary value at infinity in the open case).

Despite such very strong results, it has thus far not been possible to apply these considerations to the quantization program for gravitation. The necessity of employing a coordinate condition such as equation (1.3), while unpleasant, may not be disastrous. However, one must determine the extent to which the class of space-times are restricted by the requirement that surfaces defined by equation (1.3) are everywhere spacelike, and either closed or asymptotically flat. A more serious difficulty is engendered by the auxiliary nature of the function φ. That is, equation (1.7) does not provide φ as a functional over the constraint hypersurface of the general relativistic phase space, except in some nonlocal implicit sense.

In the present paper we shall implement a conformal transformation canonically in such a manner that the spatial constraint, equation (1.1), is transformed into equation (1.4). The remaining constraint, equation (1.2), will, a fortiori, be transformed into the York–O'Murchadha form, equation (1.7), where now the conformal factor is an explicit function of the canonical variables rather than being an ad hoc auxiliary function. It will appear that, by this device, no coordinate condition need be imposed. Although, in a precise sense this is in fact true, we will surprisingly recover the York coordinate condition, equation (1.3), when we insist upon writing the spatial constraints in the manifestly covariant form of equation (1.4). We shall finally sketch a procedure to solve the constraints which does not require the imposition of coordinate conditions.

2. CANONICAL TRANSFORMATION

Let us consider a canonical transformation from the pair g_{ij}, p^{ij} to the pair γ_{ij}, π^{ij} generated by the functional

$$A \equiv \alpha^{4\alpha^2/(9\beta^2-4\alpha^2)} \int (g_{mn}\pi^{mn})^{\alpha} g^{\beta/2}\, d^3x \tag{2.1}$$

where the constants α and β are at our disposal. In our earlier attempt to simplify the constraints (Komar, 1971) by means of a canonical transformation it was pointed out that when $\alpha + \beta = 1$ the spatial constraints, equation (1.1), remain form invariant. As it is our intention now to alter the form of the spatial constraints it will be necessary to have $\alpha + \beta \neq 1$. This will have as a consequence that the new canonical variables γ_{ij}, π^{ij} will have altered density weight, which we shall have to take into consideration when defining their covariant derivatives.

Proceeding in the usual fashion we determine the canonical transformation via

$$\begin{aligned} p^{ij} &= \frac{\delta A}{\delta g_{ij}} \\ &= \alpha^{9\beta^2/(9\beta^2-4\alpha^2)}(g_{mn}\pi^{mn})^{\alpha-1} g^{\beta/2}\pi^{ij} + \frac{\beta}{2}\,\alpha^{4\alpha^2/(9\beta^2-4\alpha^2)}(g_{mn}\pi^{mn})^{\alpha} g^{\beta/2} g^{ij} \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \gamma_{ij} &= \frac{\delta A}{\delta \pi^{ij}} \\ &= \alpha^{9\beta^2/(9\beta^2-4\alpha^2)}(g_{mn}\pi^{mn})^{\alpha-1} g^{\beta/2} g_{ij} \end{aligned} \tag{2.3}$$

Inverting equations (2.2) and (2.3) we find

$$g_{ij} = \gamma^{-\beta/(2\alpha+3\beta)}\pi^{2(1-\alpha)/(2\alpha+3\beta)}\gamma_{ij} \tag{2.4}$$

and

$$p_{ij} = \gamma^{\beta/(2\alpha+3\beta)}\pi^{2(\alpha-1)/(2\alpha+3\beta)}\left(\pi^{ij} + \frac{\beta}{2\alpha}\,\pi\gamma^{ij}\right) \tag{2.5}$$

where $\gamma \equiv \det \gamma_{ij}$, $\pi \equiv \gamma_{ij}\pi^{ij}$, and γ^{ij} is the reciprocal of γ_{ij}. We note that with regard to the spatial metric we have performed a conformal transformation. However, unlike York, we do not transform the momenta as in equation (1.6). The following expressions, which are an immediate consequence of equations (2.4) and (2.5), will be found to be important:

$$g^{ij} = \gamma^{\beta/(2\alpha+3\beta)}\pi^{2(\alpha-1)/(2\alpha+3\beta)}\gamma^{ij} \tag{2.6}$$

$$g = \gamma^{2\alpha/(2\alpha+3\beta)}\pi^{6(1-\alpha)/(2\alpha+3\beta)} \tag{2.7}$$

$$p = \frac{2\alpha + 3\beta}{2\alpha}\pi \tag{2.8}$$

We also note that the canonical transformation inverse to that of equations (2.4) and (2.5) is given by

$$\gamma_{ij} = \left(\frac{2\alpha}{2\alpha + 3\beta}\right)^{(\alpha-1)/\alpha} g^{\beta/2\alpha}p^{(\alpha-1)/\alpha}g_{ij} \tag{2.9}$$

and

$$\pi^{ij} = \left(\frac{2\alpha}{2\alpha + 3\beta}\right)^{(1-\alpha)/\alpha} g^{-\beta/2\alpha}p^{(1-\alpha)/\alpha}\left(p^{ij} - \frac{\beta}{2\alpha + 3\beta}pg^{ij}\right) \tag{2.10}$$

We see by inspection that the above canonical transformation will be nonsingular provided

$$\alpha(2\alpha + 3\beta) \neq 0 \tag{2.11}$$

For the standard canonical variables the density weights are

$$\omega(g_{ij}) = 0 \tag{2.12a}$$

$$\omega(p^{ij}) = 1 \tag{2.12b}$$

$$\omega(p) = 1 \tag{2.12c}$$

$$\omega(g) = 2 \tag{2.12d}$$

Using equations (2.12) we readily find from equations (2.9) and (2.10) the density weights for the canonically transformed variables, thus:

$$\omega(\gamma_{ij}) = \frac{\alpha + \beta - 1}{\alpha} \tag{2.13a}$$

$$\omega(\pi^{ij}) = \frac{1 - \beta}{\alpha} \tag{2.13b}$$

$$\omega(\pi) = 1 \tag{2.13c}$$

$$\omega(\gamma) = \frac{5\alpha + 3\beta - 3}{\alpha} \tag{2.13d}$$

3. THE TRANSFORMED CONSTRAINTS

In order to determine the canonically transformed constraints it is sufficient to substitute equations (2.4)–(2.8) into equations (1.1) and (1.2). For the spatial constraint we obtain

$$\mathscr{H}^s = -2\gamma^{\beta/(2\alpha+3\beta)}\pi^{2(\alpha-1)/(2\alpha+3\beta)}\left[\pi^{sm}_{,m} + \pi^{mn}\gamma^s_{mn} + \frac{\alpha+\beta-1}{2\alpha}\gamma^{sm}\pi_{,m}\right] \tag{3.1}$$

where γ^s_{mn} is defined to be the Christoffel symbol constructed with respect to γ_{ij}; that is,

$$\gamma^s_{mn} \equiv \tfrac{1}{2}\gamma^{sr}(\gamma_{rm,n} + \gamma_{rn,m} - \gamma_{mn,r}) \tag{3.2}$$

We must emphasize, however, in view of equation (2.13a), that neither does γ^s_{mn} transform as an affine connection, nor is γ_{st} covariantly constant should we attempt to define covariant differentiation by means of γ^s_{mn}. Thus, although one can confirm by direct computation that $\mathscr{H}^s$, as defined by equation (3.1), is a spatial vector density, the explicit expression given therein is not manifestly so. In order to exhibit $\mathscr{H}^s$ in a manifestly covariant form, let us introduce a transformed affine connection Λ^s_{mn} so defined that the tensor density γ_{ij} of weight $(\alpha + \beta - 1)/\alpha$ is covariantly constant. Thus,

$$0 = \gamma_{ij|k} = \gamma_{ij,k} - \gamma_{im}\Lambda^m_{jk} - \gamma_{mj}\Lambda^m_{ik} - \frac{\alpha+\beta-1}{\alpha}\gamma_{ij}\Lambda^m_{mk} \tag{3.3}$$

from which we conclude

$$\Lambda^i_{jk} = \gamma^i_{jk} - \frac{\alpha+\beta-1}{5\alpha+3\beta-3}(\delta_k{}^i\gamma^m_{mj} + \delta_j{}^i\gamma^m_{mk} - \gamma^{in}\gamma_{jk}\gamma^m_{mn}) \tag{3.4}$$

Employing this new affine connection Λ^i_{jk} in the definition of covariant differentiation, the spatial constraint, equation (3.1), may now be written in the manifestly covariant form

$$\mathscr{H}^s = -2\gamma^{\beta/(2\alpha+3\beta)}\pi^{2(\alpha-1)/(2\alpha+3\beta)}\left[\pi^{sm} + \frac{\alpha+\beta-1}{2\alpha}\gamma^{sm}\pi\right]_{|m} \tag{3.5}$$

The temporal constraint $\mathscr{H}_L$ has an exceedingly intricate, and not very illuminating, form in terms of the new canonical variables. For a particular simplifying choice of α and β it will be given in a later section. In the manifestly covariant notation, the form of $\mathscr{H}_L$ becomes considerably more tractable. Thus, if we define

$$\rho^{ij} \equiv \pi^{ij} + \frac{\alpha+\beta-1}{2\alpha}\gamma^{ij}\pi \tag{3.6}$$

and

$$\varphi \equiv \pi^{-(\alpha-1)/2(2\alpha+3\beta)} \tag{3.7}$$

we find

$$\mathscr{H}_L = \gamma^{-\alpha/(2\alpha+3\beta)}\varphi^{-6}$$
$$\times \left(\rho_{mn}\rho^{mn} - \frac{(18\alpha^2 + 24\alpha\beta + 9\beta^2 - 20\alpha - 12\beta + 6)}{8\alpha^2}\varphi^{-4(2\alpha+3\beta)/(\alpha-1)}\right)$$
$$+ \gamma^{\alpha(3\alpha+3\beta-1)/(5\alpha+3\beta-3)(2\alpha+3\beta)}\varphi(\varphi P - \varphi|_m^m) \tag{3.8}$$

where P is the Ricci scalar associated with Λ^i_{jk}.

4. THE YORK–O'MURCHADHA EQUATIONS

In order to facilitate a comparison with the work of York and O'Murchadha it will be convenient in this section to introduce the tensor field (of zero weight) $\bar{g}_{ij}$ associated with the tensor density which is our canonical configuration variable γ_{ij}. Thus we define

$$\bar{g}_{ij} \equiv \gamma^{-2\alpha(\alpha+\beta-1)/(5\alpha+3\beta-3)}\gamma_{ij} \tag{4.1}$$

We note that with respect to the affine connection Λ^i_{jk}, $\bar{g}_{ij}$ is covariantly constant. Thus Λ^i_{jk} may be regarded as the Christoffel symbol formed from $\bar{g}_{ij}$, and P is proportional to the Ricci scalar formed from $\bar{g}_{ij}$. Denoting the determinant of $\bar{g}_{ij}$ by $\bar{g}$ we obtain from equation (4.1)

$$\gamma = \bar{g}^{(5\alpha+3\beta-3)/2\alpha} \tag{4.2}$$

If we substitute equation (4.2) into equation (3.8) we find the slight modification

$$\mathscr{H}_L = \bar{g}^{-(5\alpha+3\beta-3)/2(2\alpha+3\beta)}\varphi^{-6}$$
$$\times \left(\rho^{mn}\rho_{mn} - \frac{(18\alpha^2 + 24\alpha\beta + 9\beta^2 - 20\alpha - 12\beta + 6)}{8\alpha^2}\varphi^{-4(2\alpha+3\beta)/(\alpha-1)}\right)$$
$$+ \bar{g}^{(3\alpha+3\beta-1)/2(2\alpha+3\beta)}\varphi(\varphi P - 8\varphi|_m^m) \tag{4.3}$$

In addition, with the notation of equation (3.6), the remaining constraints, equation (3.5), may be written

$$\mathscr{H}^s = -2\bar{g}^{\beta(5\alpha+3\beta-3)/2\alpha(2\alpha+3\beta)}\pi^{2(\alpha-1)/(2\alpha+3\beta)}\rho^{sm}{}_{|m} \tag{4.4}$$

We have had the constants α and β at our disposal until this point. Referring to equation (3.6), we see that if we select α and β such that

$$\frac{\alpha + \beta - 1}{2\alpha} = -\tfrac{1}{3} \tag{4.5}$$

or equivalently

$$5\alpha + 3\beta - 3 = 0 \tag{4.6}$$

then the spatial constraint $\mathscr{H}^s = 0$ has precisely the form of the York equation (1.4), albeit in canonically transformed variables. Furthermore, in view of the fact that equation (4.6) is consistent with the inequality (2.11) the associated canonical transformation is nonsingular. With α and β so chosen, we obtain from the temporal constraint $\mathscr{H}_L = 0$, via equation (4.3),

$$8\varphi|_m^m = P\varphi + \bar{g}^{-1/3}\rho^{mn}\rho_{mn}\varphi^{-7} - \frac{3(\alpha - 1)^2}{8\alpha^2}\bar{g}^{-1/3}\varphi^5 \tag{4.7}$$

which we recognize to be the York–O'Murchadha equation (1.7). Now, however, via equations (3.7) and (4.6), φ is the known local function of the canonical variables given by

$$\varphi = \pi^{1/6} \tag{4.8}$$

The analysis of equation (4.7) given by York and O'Murchadha demonstrates the degree to which π is determined uniquely by the temporal constraint. In the present situation, we have obtained these relations by the implementation of a nonsingular canonical transformation, rather than by the introduction of an ad hoc conformal factor. It also would appear that no coordinate condition was employed to obtain the York–O'Murchadha form for the constraints. However, this question requires closer scrutiny.

5. THE EFFECTIVE COORDINATE CONDITION

The essential reason for the imposition of equation (4.6) is that it renders ρ_{ij} trace-free, thereby decoupling the spatial constraint equations $\mathscr{H}^s = 0$ from the temporal constraint equation $\mathscr{H}_L = 0$. Thus, one is able to solve for ρ_{ij} independently of π, and then employ equation (4.7) to determine π. However, if we refer to equation (2.13d) we find that equation (4.6) has as a consequence

$$\omega(\gamma) = 0 \tag{5.1}$$

Thus γ has become a scalar, rather than a density. It follows that there can only exist an affine connection for which γ_{ij} is covariantly constant provided γ is a constant. The explicit expression for Λ^i_{jk}, equation (3.4), is seen to become singular unless $\gamma^m_{mk} = 0$. However, setting γ equal to a constant is precisely equivalent to York's coordinate condition, equation (1.3). In fact, substituting equations (2.7), (2.8), and (4.6) into the right-hand side of equation (1.3) we find

$$t = \frac{1 - \alpha}{\alpha}\gamma^{-\alpha/3(1-\alpha)} \tag{5.2}$$

Note that our condition of nonsingularity, equation (2.11), is now

$$\alpha(1 - \alpha) \neq 0 \tag{5.3}$$

If we attempt to employ the auxiliary metric tensor $\bar{g}_{ij}$, we find from equation (4.2) that equation (4.6) imposes the effective coordinate condition $\gamma = 1$. However, despite appearances, we must emphasize that, provided equation (5.3) is satisfied, the canonical transformation that we have performed is not singular. It is our effort to write the constraints in a manifestly covariant form, thereby decoupling the spatial and temporal constraints, that has introduced this singular behavior.

In order to clarify this latter point, and to discover what our canonical transformation has accomplished before the introduction of a coordinate condition, let us reexamine the form that the constraints take in terms of the canonical variables γ_{ij}, π^{ij}. For simplicity, consistent with equation (4.6), we shall consider the case where

$$\alpha = \tfrac{3}{5}, \qquad \beta = 0 \tag{5.4}$$

By direct substitution into equation (3.1), the spatial constraint now yields

$$\mathscr{H}^s = \pi^{-2/3}(\rho^{sm}_{,m} + \gamma^s_{mn} p^{mn} - \tfrac{1}{3}\pi\gamma^{sn}\gamma^m_{mn}) = 0 \tag{5.5}$$

where, via equation (3.6),

$$\rho^{ij} = \pi^{ij} - \tfrac{1}{3}\pi\gamma^{ij} \tag{5.6}$$

Similarly, by direct substitution of equations (2.4)–(2.8), into equation (1.2), and employing the definition of equation (4.8) we find for the temporal constraint

$$\mathscr{H}_L = \gamma^{1/2}\varphi[\mathscr{P}\varphi - 8\gamma^{mn}(\varphi_{,mn} - \gamma^r_{mn}\varphi_{,r}) + \gamma^{-1}\rho^{mn}\rho_{mn}\varphi^{-7} - \tfrac{1}{6}\gamma^{-1}\varphi^5] = 0 \tag{5.7}$$

where $\mathscr{P}$ is the expression for the Ricci curvature scalar constructed as if γ_{ij} were a metric tensor. We see that the above expressions for $\mathscr{H}^s$ and H_L are perfectly regular, as anticipated. In fact, equation (5.7) retains the form of the York–O'Murchadha equation. From an analytic point of view, we can neglect the density properties of φ and γ_{rs}, and regard them, respectively, as a scalar and a symmetric tensor, in which event equation (5.7) may be written essentially identical to equation (1.7).

On the other hand, our canonical transformation does not succeed in decoupling the equation for ρ^{ij}, equation (5.5), from equation (5.7). Its accomplishment is to make the spatial constraints algebraic in π. The decoupling is evidently accomplished only by the coordinate condition $\gamma^m_{ms} = 0$ or equivalently $\gamma = \text{const}$.

6. CONCLUDING REMARKS

We have attempted to decouple the spatial and temporal constraints of general relativity without employing coordinate conditions. In order to accomplish this we implemented a York-like conformal transformation canonically; however, we discovered that without a coordinate condition the constraints remained coupled. Our canonical transformation did succeed in simplifying the constraints to the extent that the temporal constraint took the form of the well-analyzed York–O'Murchadha equation, while the spatial constraints became algebraic in π. The extent to which this latter property proves significant remains to be explored.

The inhomogeneous equation for ρ^{ij}, equation (5.5), may easily be solved by York's method (York, 1973) of separating traceless tensors into their longitudinal and transverse parts. More explicitly, from an analytic point of view we may introduce notation that is adapted to treating ρ^{ij} as a tensor density of weight 1 and γ^s_{mn} as a proper affine connection. In that event we may obtain from equation (5.5) the inhomogeneous equation for ρ^{ij}

$$\rho^{sm}{}_{|m} = \tfrac{1}{3}\pi\gamma^{sm}\gamma^n_{nm} \tag{6.1}$$

We recognize that, owing to the fact that the notation for covariant differentiations now refers to the connection γ^s_{mn}, which, as we have noted, does not transform as an affine connection, equation (6.1) is not manifestly covariant, although it is a true vector density relationship. Following York, this equation may be solved by writing

$$\rho^{ij} = \rho^{ij}_{TT} + \rho^{ij}_L \tag{6.2}$$

where

$$\rho^{im}_{TT|m} \equiv 0 \tag{6.3}$$

and

$$\rho^{ij}_L \equiv \xi^{i|j} + \xi^{j|i} - \tfrac{2}{3}\gamma^{ij}\xi^m{}_{|n} \tag{6.4}$$

Substitution of the above three relations into equation (6.1) yields for ρ^{ij}_L the unique (York, 1973) solution

$$\rho^{ij}_L = \tfrac{1}{3}[(D^{-1}\pi\gamma^{im}\gamma^n_{nm})^{|j} + (D^{-1}\pi\gamma^{jm}\gamma^n_{nm})^{|i} - \tfrac{2}{3}\gamma^{ij}(D^{-1}\pi\gamma^{rm}\gamma^n_{nm})_{|r}] \tag{6.5}$$

where D is the linear differential operator defined by

$$D\xi^i \equiv \xi^{i|m}{}_m + \xi^{m|i}{}_m - \tfrac{2}{3}\xi^m{}_{|m}{}^{|i} \tag{6.6}$$

Thus ρ^{ij}_L is given as an explicit (nonlocal) functional of π, while the equation for ρ^{ij}_{TT}, equation (6.3), is completely decoupled from the temporal constraint, equation (5.7), and can be solved in full generality. Substituting the resulting expression for ρ^{ij} into equation (5.7), we obtain an equation for φ, or equivalently for π, without employing coordinate conditions. It requires further investigation to determine to what extent the uniqueness result of York and O'Murchadha is altered by the fact that the term $\rho^{mn}\rho_{mn}$ in the temporal constraint is now a functional of φ.

It is at this point that the property of the inhomogeneous term in the spatial constraint being proportional to π may be of significance. For it is important to the proof of uniqueness that the inequality

$$\rho^{mn}\rho_{mn} > 0 \tag{6.7}$$

be satisfied (except possibly on a set of measure zero). Since the spatial metric is positive definite it is evident that we have $\rho^{mn}\rho_{mn} \geqslant 0$ and further that

$$\rho^{mn}\rho_{mn} = 0 \Leftrightarrow \rho^{ij} = 0 \tag{6.8}$$

However, should this equality be valid in a finite region it would then follow from the spatial constraint, or equivalently, from equation (6.1), that

$$\pi\gamma^{sm}\gamma^{n}_{nm} = 0 \tag{6.9}$$

Thus, in the generic case (that is, in the absence of coordinate conditions such as $\gamma = \text{const}$), we are assured that either the inequality equation (6.7) is satisfied or $\pi^{ij} = 0$ is the (singular) solution of the constraints. We do not wish to imply that in the present circumstance this property is sufficient to establish the uniqueness of the solutions of equation (5.7), but merely that the prospect of establishing such a result is by no means hopeless.

As a final observation, we would like to note that, by methods similar to those employed in this paper, it is possible to construct more general canonical transformations that are algebraic but not conformal. In every example of this kind that we have examined, the condition that decoupled the spatial and temporal constraint equations invariably implied an effective coordinate condition quite similar to that of equation (1.3). It is not clear, however, whether this feature is coincidental or has some subtler significance.

REFERENCES

Dirac, P. A. M. (1959). *Physical Review*, **114**, 924.
Komar, A. (1971). *Physical Review D*, **4**, 923.
O'Murchadha, N., and York, J. W. (1973). *Journal of Mathematical Physics*, **14**, 1551.
York, J. W. (1972). *Physical Review Letters*, **28**, 1082.
York, J. W. (1973). *Journal of Mathematical Physics*, **14**, 456.